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TWO-FREQUENCY RESONANT OSCILLATIONS OF CONSERVATIVE SYSTEMS

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Two-frequency oscillations of a conservative system with n degrees of freedom are studied. The problem is reduced to investigate canonical systems describing resonance phenomena. Single-frequency and multi-frequency oscillations were studied earlier in [1 – 3].

1. Let us consider a conservative system with n degrees of freedom which has a stable state of equilibrium and executes relatively small motions in the neighborhood of this state. The differential equations of motion of the system have the form

$$\sum_{k=1}^{n}(a_{ik}q_k^{\cdot\cdot}+c_{ik}q_k)=-\sum_{k,j=1}^{n}\left[a_{ik}^{(j)}(q_k\ddot{q}_j+\frac{1}{2}\dot{q}_k\dot{q}_j)+\frac{1}{2}c_{ik}^{(j)}q_kq_j\right]-\tag{1.1}$$
$$\sum_{k,j,s=1}^{n}\left[\frac{1}{2}a_{ik}^{(js)}(q_kq_jq_s^{\cdot\cdot}+q_k\dot{q}_j\dot{q}_s)+\frac{1}{6}c_{ik}^{(js)}q_kq_jq_s\right]-\ldots\ (i=1,2,\ldots n)$$

Let the system undergo two-frequency oscillations of frequencies ω_1 and $\omega_2(\omega_2\geqslant\omega_1)$. We shall consider the two-frequency resonant solutions of the system (1.1). We define the degree of the resonance terms in the right-hand sides of (1.1), as the resonance rank. The ratio ω_2/ω_1 for these terms is essential.

2. Consider at first a second order resonance. We set in (1.1) $q_k = \varepsilon \bar{q}_k$ where ε is a small positive parameter, omit the superscript bar and introduce the dimensionless time $\tau = \omega_1 t$. Then from (1.1) we obtain

$$\sum_{k=1}^{p} (a_{ik} q_k'' + c_{ik} q_k) = -\varepsilon \sum_{k,j=1}^{n} \left[a_{ik}^{(j)} \left(q_k q_j'' + \frac{1}{2} q_k' q_j' \right) + \frac{1}{2} c_{ik}^{(j)} q_k q_j \right] \qquad (2.1)$$

$$(i = 1, 2, \ldots n)$$

$$\bar{c}_{ik} = \frac{c_{ik}}{\omega_1^2}, \qquad c_{ik}^{j} = \frac{c_{ik}^{(j)}}{\omega_1^2}, \qquad c_{ik}^{(js)} = \frac{c_{ik}^{(js)}}{\omega_1^2}, \ldots$$

where the terms of order higher than second are discarded. We set $\omega_2 / \omega_1 = \beta$ ($\beta \geqslant 1$) and seek the two-frequency solution of (2.1) in the form

$$q_k = L_k^{(1)} B \cos(\tau - \psi) + L_k^{(2)} A \cos(\beta\tau - \varphi) + \varepsilon q_{k1} + \varepsilon^2 q_{k2} + \cdots \qquad (2.2)$$

$$(k = 1, 2, \ldots n)$$

where A, B, φ and ψ are slowly varying functions of τ, and q_{k1}, $q_{k2}, \cdots$ are additive correction terms which can be expressed uniquely in terms of A, B, φ, ψ and τ. The normal functions $L_k^{(j)}$ (k, $j = 1, 2, \ldots n$) are obtained from the systems of algebraic equations

$$\sum_{k=1}^{n} (-a_{ik}\omega_j^2 + c_{ik}) L_k^{(j)} = 0 \qquad (i, j = 1, 2, \ldots n)$$

and possess orthogonal properties, i.e.

$$\sum_{i,k=1}^{n} a_{ik} L_i^{(j)} L_k^{(s)} = 0, \qquad \sum_{i,k=1}^{n} c_{ik} L_i^{(j)} L_k^{(s)} = 0 \qquad (j \neq s)$$

Under the assumptions made, all natural frequencies ω_j ($j = 1, 2, \ldots n$) are rational numbers.

Let us now substitute (2.2) into (2.1), multiply the i-th equation by $L_i^{(1)}$, add all equations together, and again multiply the i-th equation by $L_i^{(2)}$ and add all equations. This yields

$$(B'' + 2B\psi' - B\psi'^2)\cos(\tau - \psi) + (B\psi'' - 2B' + 2B'\psi')\sin(\tau - \psi) + \qquad (2.3)$$

$$\varepsilon \frac{1}{m_1} \sum_{i,k=1}^{n} L_i^{(1)} (a_{ik} q_{k1}'' + \bar{c}_{ik} q_{k1}) = -\varepsilon F_1$$

$$(A'' + 2\beta A\varphi' - A\varphi'^2)\cos(\beta\tau - \varphi) + (A\varphi'' - 2\beta A' + 2A'\varphi')\sin(\beta\tau - \varphi) +$$

$$\varepsilon \frac{1}{m_2} \sum_{i,k=1}^{n} L_i^{(2)} (a_{ik} q_{k1}'' + \bar{c}_{ik} q_{k1}) = -\varepsilon F_2$$

where

$$F_r = \frac{1}{m_r} \left\{ \frac{1}{4} [(g_{11}^{(r)} - h_{11}^{(r)}) B^2 + (g_{22}^{(r)} - \beta^2 h_{22}^{(r)}) A^2] + \frac{1}{4} (g_{11}^{(r)} - 3h_{11}^{(r)}) B^2 \cos(2\tau - 2\psi) + \right.$$

$$\frac{1}{4} (g_{22}^{(r)} - 3\beta^2 h_{22}^{(r)}) A^2 \cos(2\beta\tau - 2\varphi) +$$

$$\frac{1}{2} [g_{12}^{(r)} - h_{12}^{(r)} (1 - \beta + \beta^2)] AB \cos[(\beta - 1)\tau - \varphi + \psi] +$$

$$\left. \frac{1}{2} [g_{12}^{(r)} - h_{12}^{(r)} (1 + \beta + \beta^2)] AB \cos[(\beta + 1)\tau - \varphi - \psi] \right\} \quad (r = 1, 2)$$

$$m_r = \sum_{i,k=1}^{n} a_{ik} L_i^{(r)} L_k^{(r)} > 0$$

$$h_{sq}^{(r)} = \sum_{i,k,j=1}^{n} a_{ik}^{(j)} L_i^{(s)} L_k^{(q)} L_j^{(r)}, \qquad g_{sq}^{(r)} = \sum_{i,k,j=1}^{n} \bar{c}_{ik}^{(j)} L_i^{(s)} L_k^{(q)} L_j^{(r)}$$

From the latter expressions it follows that

$$h_{sq}^{(r)} = h_{sr}^{(q)} = \ldots, \qquad g_{sq}^{(r)} = q_{sr}^{(q)} = \ldots$$

i.e. the corresponding coefficients $h_{sq}^{(r)}$ and $g_{sq}^{(r)}$ with the same indices are equal to each other irrespective of the distribution of indices. In order to solve the system (2.3) we shall utilize a corollary [4] to the existing asymptotic methods in the theory of non-linear oscillations. By analyzing the system (2.3) we come to the following theorem.

T h e o r e m 1. The second rank resonance is possible only when $\beta \approx 2$.

Let $\beta \approx 2$. Then, using the trigonometric identities for $\cos(\tau - \psi + \lambda)$ and $\cos(\beta\tau - \varphi - \lambda)$ where $\lambda = (\beta - 2)\tau - \varphi + 2\psi$, and equating the corresponding terms in (2.3), we obtain the variational system

$$B'' + 2B\psi' - B\psi'^2 = -\varepsilon \frac{1}{2m_1} [g_{12}^{(1)} - h_{12}^{(1)} (1 - \beta + \beta^2)] AB \cos\lambda \qquad (2.4)$$

$$B\psi'' - 2B' + 2B'\psi' = \varepsilon \frac{1}{2m_1} [g_{12}^{(1)} - h_{12}^{(1)} (1 - \beta + \beta^2)] AB \sin\lambda$$

$$A'' + 2\beta A\varphi' - A'^2 = -\varepsilon \frac{1}{4m_2} (q_{11}^{(2)} - 3h_{11}^{(2)}) B^2 \cos\lambda$$

$$A\varphi'' - 2\beta A' + 2'\varphi' = -\varepsilon \frac{1}{4m_2} (g_{11}^{(2)} - 3h_{11}^{(2)}) B^2 \sin\lambda$$

Since we seek the solution of (2.3) only in its first approximation, we shall not set up the equations for determining the additive corrections. Assuming that $\beta \approx 2$, we find from (2.4), with the accuracy of up to the terms of first order of smallness in ε, the abridged van der Pol system in the variables A, B, φ and ψ. The first and third equations of (2.4) yield the integral

$$\sigma^2 A^2 + B^2 = \sigma^2 \varkappa^2 \qquad (\sigma^2 = 4\, m_2 / m_1) \qquad (2.5)$$

where $\varkappa^2$ is a constant of integration proportional to the total energy of the mechanical system. The relation (2.5) corresponds to the energy integral and connects the amplitudes A and B for each instant of time. From (2.5) it follows that if one amplitude decreases, the other increases. From (2.4) taking into account (2.5), we obtain the following autonomous system in the variables A^0 and λ

$$dA^\circ/du = (1 - A^{\circ 2}) \sin\lambda, \quad d\lambda/du = 2m + \frac{1}{A^\circ} (1 - A^{\circ 2}) \cos\lambda \qquad (2.6)$$

where

$$u = \varepsilon\tau \frac{g_{11}^{(2)} - 3h_{11}^{(2)}}{4m_1} \varkappa, \qquad A^\circ = \frac{A}{\varkappa}, \qquad 2m = \frac{\beta - 2}{\varepsilon} \frac{4m_1}{(g_{11}^{(2)} - 3h_{11}^{(2)})\varkappa}$$

We cannot substitute $\beta \approx 2$ into the expression for m, since we assume that the mistuning $\beta - 2$ is of the order of smallness ε. We shall call the system of the form (2.6) relative to the amplitude A° or $B^0 = B / (\sigma\varkappa)$ and the variable λ (including the phases φ and ψ), with a minimum number of parameters, the canonical system for the resonance in question. Eliminating u from (2.6) and integrating, we obtain

$$mA^{\circ 2} + A^\circ (1 - A^{\circ 2}) \cos\lambda = c_0 \qquad (2.7)$$

where c_0 is the constant of integration.

Investigation of the system (2.6) on the XY phase plane, for which $X = A^\circ \cos\lambda$ and $Y = A^\circ \sin\lambda$, i.e. A° and λ are polar coordinates, yields a complete picture of all possible motions of the mechanical system. The phase trajectories of the system (2.6) are given by the expression (2.7) and they are all symmetrical with respect to the X-axis. In view of (2.5), all real trajectories lie on the boundary of, or within the circle $A^\circ = 1$. The canonical system (2.6) was studied in detail in [5, 6] while investigating the resonant motions of specific mechanical systems with two degrees of freedom.

3. Let us consider a third rank resonance. Let the mechanical system under consideration possess an internal symmetry. Let also the kinetic and potential energies be symmetrical with respect to the generalized coordinates q_i $(i = 1, 2, \ldots n)$. Then $a_{ik}^{(j)} = c_{ik}^{(j)} = 0$ and the system (1.1) contains only terms of the odd power. In absence of such symmetry, we obtain for the third order resonance the same basic results, albeit after more lengthy computations as they have to be carried out to the second approximation.

Let us now make the substitution $q_k = \varepsilon^{1/2}\bar{q}_k$ in (1.1), again discard the superscript bar and introduce the dimensionless time $\tau = \omega_1 t$. This yields the system

$$\sum_{k=1}^{n} (a_{ik}q_k'' + \bar{c}_{ik}q_k) = -\varepsilon \sum_{k,j,s=1}^{n} \Big[\frac{1}{2} a_{ik}^{(js)} (q_k q_j q_s'' + q_k q_j' q_s') + \frac{1}{6}\bar{c}_{ik}^{(js)} q_k q_j q_s\Big] \quad (i = 1, 2, \ldots n) \qquad (3.1)$$

We seek the solution of (3.1) in the form (2.2). Analogous transformations again yield (2.3), but now we have

$$F_r = \frac{1}{m_r}\Big\{\frac{1}{8}[(g_{11}^{(2r)} - 2h_{11}^{(1r)}) B^3 + 2(g_{12}^{(2r)} - h_{12}^{(2r)}(\beta^2 + 1)) A^2B] \cos(\tau - \psi) +$$
$$\frac{1}{8}[(g_{22}^{(2r)} - 2\beta^2 h_{22}^{(2r)}) A^3 + 2(g_{12}^{(1r)} - h_{12}^{(1r)}(1 + \beta^2)) AB^2] \cos(\beta\tau - \varphi) +$$
$$\frac{1}{24}(g_{11}^{(1r)} - 6h_{11}^{(1r)}) B^3 \cos(3\tau - 3\psi) + \frac{1}{24}(g_{22}^{(2r)} - 6\beta^2 h_{22}^{(2r)}) A^3 \cos(3\beta\tau - 3\varphi) +$$
$$\frac{1}{8}[g_{12}^{(1r)} - h_{12}^{(1r)}(\beta^2 - 2\beta + 3)] AB^2 \cos[(\beta - 2)\tau - \varphi + 2\psi] +$$
$$\frac{1}{8}[g_{12}^{(1r)} - h_{12}^{(1r)}(\beta^2 + 2\beta + 3)] AB^2 \cos[(\beta + 2)\tau - \varphi - 2\psi] +$$
$$\frac{1}{8}[g_{22}^{(1r)} - h_{22}^{(1r)}(1 - 2\beta + 3\beta^2)] A^2B \cos[(2\beta - 1)\tau - 2\varphi + \psi] +$$
$$\frac{1}{8}[g_{22}^{(1r)} - h_{22}^{(1r)}(1 + 2\beta + 3\beta^2)] A^2B \cos[(2\beta + 1)\tau - 2\varphi - \psi]\Big\} \quad (r = 1, 2)$$

$$h_{rt}^{(pq)} = \sum_{i,k,j,s=1}^{n} a_{ik}^{(js)} L_i^{(r)} L_k^{(t)} L_j^{(p)} L_s^{(q)}, \qquad g_{rt}^{(pq)} = \sum_{i,k,j,s=1}^{n} \bar{c}_{ik}^{(js)} L_i^{(r)} L_k^{(t)} L_j^{(p)} L_s^{(q)}$$

As before, we have

$$h_{r}^{(pq)} = h_{rq}^{(pt)} = \ldots, \qquad g_{rt}^{(pq)} = g_{rq}^{(pt)} = \ldots$$

In the present case the analysis of the system (2.3) yields the following theorem.

T h e o r e m 2. Third rank resonance is possible only when $\beta \approx 1$ and $\beta \approx 3$. Let us first assume that $\beta \approx 1$. Using the identities for $\cos(\tau - \psi \pm \lambda)$, $\cos(\tau - \psi + 2\lambda)$,

$\cos(\beta\tau - \varphi \pm \lambda)$ and $\cos(\beta\tau - \varphi - 2\lambda)$, where $\lambda = (\beta - 1)\tau - \varphi + \psi$, and making use of the assumption that $\beta \approx 1$, we obtain the following abridged van der Pol system

$$\frac{dB}{\varepsilon d\tau} = -\frac{1}{16m_1}\{[(g_{12}^{(22)} - 2h_{12}^{22}) A^3 + (g_{12}^{(11)} - 2h_{12}^{(11)}) AB^2]\sin\lambda + (g_{22}^{(11)} - 2h_{22}^{(11)}) A^2B\sin 2\lambda\} \tag{3.2}$$

$$\frac{d\psi}{\varepsilon d\tau} = -\frac{1}{16m_1}\{(g_{11}^{(11)} - 2h_{11}^{(11)}) B^2 + 2(g_{22}^{(11)} - 2h_{22}^{(11)}) A^2 + (g_{22}^{(11)} - 2h_{22}^{(11)}) \times A^2\cos 2\lambda + [(g_{22}^{(12)} - 2h_{22}^{(12)})\frac{A^3}{B} + 3(g_{12}^{(11)} - 2h_{12}^{(11)}) AB]\cos\lambda\}$$

$$\frac{dA}{\varepsilon d\tau} = \frac{1}{16m_2}\{(g_{12}^{(11)} - 2h_{12}^{(11)}) B^3 + (g_{12}^{(22)} - 2h_{12}^{(22)}) A^2B\sin\lambda + (g_{22}^{(11)} - 2h_{22}^{(11)}) AB^2\sin 2\lambda\}$$

$$\frac{d\varphi}{\varepsilon d\tau} = -\frac{1}{16m_2}\left\{(g_{22}^{(12)} - 2h_{22}^{(12)}) A^2 + 2(g_{22}^{(11)} - 2h_{22}^{(11)}) B^2 + (g_{22}^{(11)} - 2h_{22}^{(11)}) \times B^2\cos 2\lambda + [(g_{12}^{(11)} - 2h_{12}^{(11)})\frac{B^3}{A} + 3(g_{12}^{(22)} - 2h_{12}^{(22)}) AB]\cos\lambda\right\}$$

The first and third equation of (3.2) again yield the integral (2.5), but now we have $\sigma^2 = m_2 / m_1$. From (3.2) and (2.5) we obtain the following canonical system for the resonance:

$$dA^\circ / \partial u = (1 - A^{\circ 2})^{3/2}\ [b(1 - A^{\circ 2}) + cA^{\circ 2}]\sin\lambda + 2A^\circ(1 - A^{\circ 2})\sin 2\lambda \tag{3.3}$$

$$\partial\lambda / du = 2m + 4aA^{\circ 2} + [b(1 - 4A^{\circ 2})(1 - A^{\circ 2})^{1/2} A^{\circ -1} + cA^\circ(3 - 4A^{\circ 2})(1 - A^{\circ 2})^{-1/2}]\cos\lambda + 2(1 - 2A^{\circ 2})\cos 2\lambda$$

where

$$u = \varepsilon\tau\frac{1}{32m_1}(g_{22}^{(11)} - 2h_{22}^{(11)})\varkappa^2, \qquad b = 2\sigma\frac{g_{12}^{(11)} - 2h_{12}^{(11)}}{g_{22}^{(11)} - 2h_{22}^{(11)}}$$

$$m = \frac{\beta - 1}{\varepsilon}\frac{16m_1}{(g_{22}^{(11)} - 2h_{22}^{(11)})\varkappa^2} - \sigma^2\frac{g_{11}^{(11)} - 2h_{11}^{(11)}}{g_{22}^{(11)} - 2^{(11)}_{22}} + 2$$

$$4a = 2\frac{g_{22}^{(22)} - 2h_{22}^{(22)}}{(g_{22}^{(11)} - 2h_{22}^{(11)})\sigma^2} + 2\sigma^2\frac{g_{11}^{(11)} - 2h_{11}^{(11)}}{g_{22}^{(11)} - 2h_{22}^{(11)}} - 8$$

$$c = \frac{2}{\sigma}\frac{g_{22}^{(12)} - 2h_{22}^{(12)}}{g_{22}^{(11)} - 2h_{22}^{(11)}}, \qquad A^\circ = \frac{A}{\varkappa}, \qquad g_{22}^{(11)} - 2h_{22}^{(11)} \neq 0$$

The system (3.3) has the following integral:

$$mA^{\circ 2} + aA^{\circ 4} + [bA^\circ(1 - A^{\circ 2})^{3/2} + cA^{\circ 3}(1 - A^{\circ 2})^{1/2}]\cos\lambda + A^{\circ 2} \times (1 - A^{\circ 2})\cos 2\lambda = c_0 \tag{3.4}$$

where c_0 is the constant of integration.

4. Let us investigate the system (3.3) on the XY phase plane ($X = A^\circ\cos\lambda$ and $Y = A^\circ\sin\lambda$, where A° and λ are polar coordinates). The phase trajectories of the system (3.3) are given by the expression (3.4) and are all symmetric with respect to the X-axis. All real trajectories lie at the boundary of, or within the circle $A^\circ = 1$.

First we find the singularities of the system (3.3). From the conditions $dA^\circ / du = d\lambda / du = 0$ we see that three types of singularities exist: points (a) on the X-axis, points

(b) on the curve

$$K\,\{b\,(1-A^{\circ 2})+cA^{\circ 2}+4A^{\circ}\,(1-A^{\circ 2})^{1/2}\cos\lambda=0\}$$

(they always appear as a pair of points distributed symmetrically relative to the X-axis) and points (c) which lie on the boundary of circle $A^{\circ}=1$. The system (3.3) was studied in detail in [7] for the case $b=c=0$.

Let us first consider the case $b\neq 0$, $c=0$. The curve K now becomes a half-ellipse. The polar angle of the points (c) is given by

$$\cos\lambda=\pm\left(\frac{1+m}{2}+a\right)^{1/2}$$

and there are four of these points, distributed symmetrically with respect to the X- and Y-axes.

The abscissas of the points (a) are found from the equation

$$m^{*}X=-2a^{*}X^{3}-\frac{1}{2}(1-4X^{2})(1-X^{2})^{1/2}\left(m^{*}=\frac{m+1}{b},\quad a^{*}=\frac{a-1)}{b}\right) \qquad (4.1)$$

From (4.1) we see that we have either one or three points (a) (except in the limiting cases). Their distribution can be obtained by solving (4.1) graphically.

Points (b) exist for m in the interval

$$(m_{\delta_1},\ m_{\delta_2}),\qquad m_{\delta_1}=[16-b^{2}(5+2a)]\,/\,(16+b^{2}),\qquad m_{\delta_2}=-1-2a.$$

while points (c) exist in the interval $(-1-2a,\ 1-2a)$.

Various typical patterns of distribution of the phase trajectories are determined by the number and type of the singularities. The system (3.3) gives, even in the case $c=0$, a large number of such typical patterns differing from each other, therefore we shall only indicate some of them. A case is possible, when three (a) and (b)-type singularities exist (see Figs. 1 and 2). Figure 1 shows the points (b) as centers, two of the points (a) as saddles and one as a center. Figure 2 shows the points (b) as saddles and points (a) as centers. Figure 3 depicts the case when all three points (a) are centers and all three (c) are saddles. Figures 4 and 5 show the cases when all possible singularities exist, three (a), two (b) and four (c). In Fig. 4 the separatrix passing through the saddle point (a) embraces two points (b), and in Fig. 5 a similar separatrix embraces the other two points (a).

Let us now consider the case when $b=0$, $c\neq 0$. In this case we replace the variable A° by B° ($B^{\circ}=B\,/\,(\sigma\varkappa)$) using the relation (2.5). Introducing $u^{\circ}=-u$ and $m^{\circ}=-m-2a$ we again obtain, in the new notation, the system (3.3) for $b\neq 0$, $c=0$ which has already been investigated. We can therefore state that in the case $b=0$, $c\neq 0$ we have exactly the same number of typical patterns of the phase trajectories as in the case $b\neq 0$, $c=0$.

The canonical systems in the variables A°, λ and B°, λ for which the relation $A^{\circ 2}+B^{\circ 2}=1$ holds, shall be called circularly conjugate. Figures 6–8 show the phase trajectories for the circularly conjugate systems, which are depicted in Fig. 1, 4 and 5, respectively. Here the singularities (c) exist in all cases and their coordinates are $A^{\circ}=1$, $\lambda=n\,/\,2$ and $3n/\,2$. We have either two or four (a)-type singularities (the points (a) include the origin of reference which is always a singularity). Points (b) now lie on a fourth order curve.

Let us consider the case $b\neq 0$, $c\neq 0$. Replacing in (3.3) A° by B°, and using (2.5) we again obtain system (3.3), although in a different notation. It can be asserted that

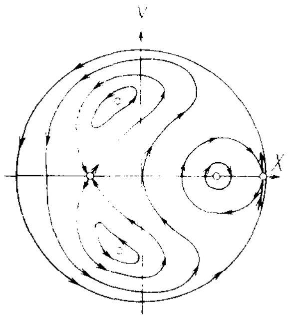

Fig. 1

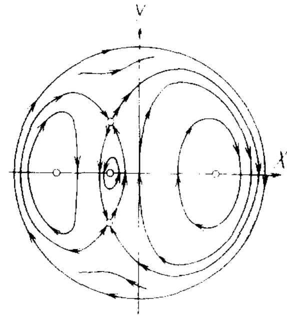

Fig. 2

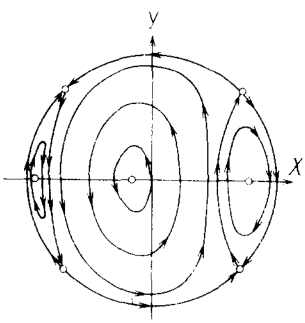

Fig. 3

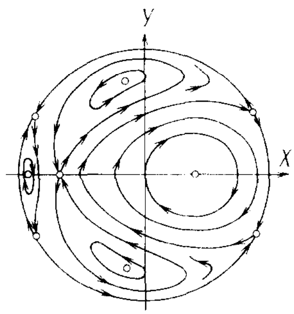

Fig. 4

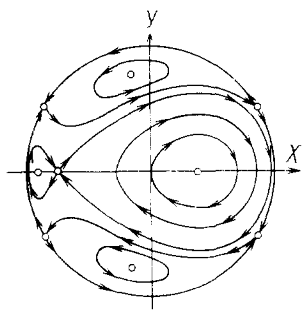

Fig. 5

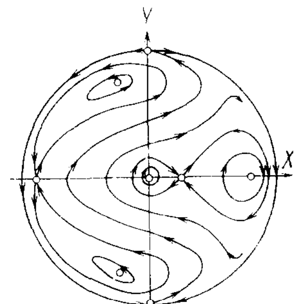

Fig. 6

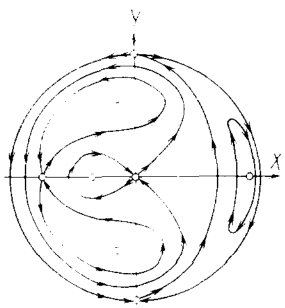

Fig. 7

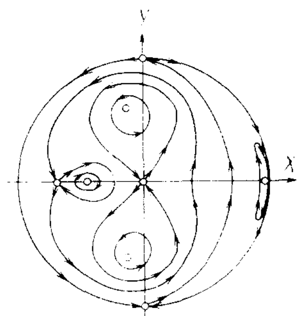

Fig. 8

the system (3.3) is therefore circularly self-conjugated. Here we encounter a large number of typical patterns of the phase trajectories due to the presence of four parameters, a, b, c and m. We note that now we have either two or four (a)-type points (but the origin of reference is no longer a singularity), and everything said previously remains valid just as in the case $b = 0$, $c \neq 0$. Therefore Figs. 6 – 8 also give a qualitative representation for this general case.

5. Let $\beta \approx 3$. Using the identities for $\cos(\tau - \psi + \lambda)$ and $\cos(\beta\tau - \varphi - \lambda)$, where $\lambda = (\beta - 3)\tau - \varphi + 3\psi$, we obtain, in analogy with the previous cases, the following canonical system for this resonance:

$$dA^\circ/du = (1 - A^{\circ 2})^{3/2} \sin\lambda \tag{5.1}$$
$$d\lambda/du = 2m + 4aA^{\circ 2} + \frac{1}{A^\circ}(1 - A^{\circ 2})^{1/2}(1 - 4A^{\circ 2})\cos\lambda$$

where

$$A^\circ = A/\varkappa, \quad u = \varepsilon\tau\frac{\sigma\varkappa^2}{16m_1}(g_{12}^{(11)} - 6h_{12}^{(11)}), \quad g_{12}^{(11)} - 6h_{12}^{(11)} \neq 0$$

$$2m = \frac{\beta - 3}{\varepsilon}\frac{16m_1}{\sigma\varkappa^2(g_{12}^{(11)} - 6h_{12}^{(11)})} + 6\frac{g_{12}^{(22)} - 10h_{12}^{(22)}}{\sigma(g_{12}^{(11)} - 6h_{12}^{(11)})} - 3\sigma\frac{g_{11}^{(11)} - 2h_{11}^{(11)}}{g_{12}^{(11)} - 6h_{12}^{(11)}}$$

$$4a = \frac{3}{\sigma^3}\frac{g_{22}^{(22)} - 18h_{22}^{(22)}}{g_{12}^{(11)} - 6h_{12}^{(11)}} - \frac{6}{\sigma}\frac{g_{11}^{(22)} - 10h_{11}^{(22)}}{g_{12}^{(11)} - 6h_{12}^{(11)}} - \frac{6}{\sigma}\frac{g_{12}^{(22)} - 10h_{12}^{(22)}}{g_{12}^{(11)} - 6h_{12}^{(11)}} + \sigma\frac{g_{11}^{(11)} - 2h_{11}^{(11)}}{g_{12}^{(11)} - 6h_{12}^{(11)}}$$

A and B are again connected by (2.5), but here $\sigma^2 = 9m_2 / m_1$. The system (5.1) has the following integral:

$$mA^{\circ 2} + aA^{\circ 4} + A^\circ(1 - A^{\circ 2})^{3/2}\cos\lambda = c_0 \tag{5.2}$$

where c_0 is the constant of integration. The only singularities existing here are of the (a)-type; there are either one or three of them (with the exception of the boundary cases), and we have relatively a small number of typical patterns of the phase trajectories.

The canonical system (5.1) was obtained and studied in [8], while solving the problem on resonant oscillations of a conservative system with two degrees of freedom.

The patterns of the phase trajectories show that motions with constant amplitudes are feasible. These motions will be periodic and will have the corresponding center-type points. Motions with periodically varying amplitudes are also possible, and the corresponding trajectories will be closed. The separatrices and saddle-type singularities correspond to the transitional (nonperiodic) changes in the amplitudes. The results obtained occur, in particular, in the resonant motions of conservative systems with two degrees of freedom.

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ON PARTICULAR SOLUTIONS OF THE PROBLEM OF MOTION OF A GYROSCOPE IN GIMBAL MOUNT

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Particular solutions of equations of motion of a heavy gyroscope in gimbal mount with the outer gimbal axis horizontal was considered in [1, 2].

Particular solutions of this problem are considered below in the case, when the axis of rotation of the outer gimbal is horizontal and the center of gravity of the gyroscope and of the casing are not lying on the axis of symmetry of the gyroscope ellipsoid of inertia but in a plane passing through the axis of symmetry perpendicularly to the axis of rotation of the inner gimbal. The latter solutions also complement each other symmetrically, similarly to those mentioned above.

The fixed system of coordinates $O\xi\eta\zeta$ is permanently attached to the axis of rotation of the outer gimbal (Fig. 1). The ξ-axis lies on the axis of rotation of that gimbal. The system of coordinate axes $Ox_2y_2z_2$ is permanently attached to the outer gimbal. The x_2- and y_2-axes coincide with the axes of rotation of the outer and inner gimbals, respectively. The system of coordinates $Ox_1y_1z_1$ is permanently attached to the casing. The y_1-axis is directed along tha casing axis of rotation and the z_1-axis along the rotor spin axis. Axes x_1, y_1 and z_1 are the principal axes of the ellipsoid of the casing inertia about the fixed point O.

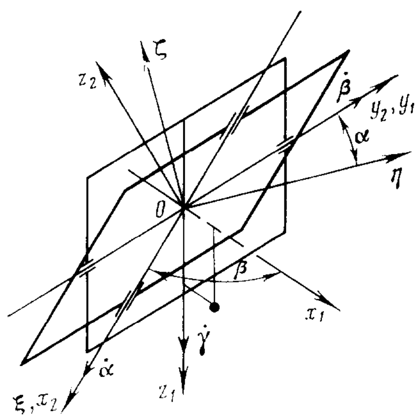

Fig. 1

Let us assume that the ellipsoid of rotor inertia about point O is the ellipsoid of rotation of the casing about the z_1-axis.

We use the following notation: α is the angle of turn of the outer gimbal; β is the angle of turn of the casing (inner gimbal), γ is the angle of turn of the rotor in the casing (angle of spin of the gyroscope about the z_1-axis); A_2 is the moment of inertia of the outer gimbal about the ξ-axis; A_1, B_1